Maths formula book

Nick Urbanik

In memory of Gunter Beck, a great teacher

Contents

Introduction

This is a collection of formulas given by my mathematics teacher, Gunter Beck, at Meadowbank Technical College, in Australia, during his teaching of four unit maths (Higher School Certificate). As he introduced each topic, he wrote the main formulas for that topic on the board, and encouraged us to write them into our own formula books. He died rather quickly from a cancer. He was my favourite teacher. After I learned of his death, and noticed that my old formula book was getting very dog eared from so many years of constant use, I decided to write this in fond memory of Gunter, using $\mathcal{AMS}$-LaTeX. There are many who miss you, Gunter.

1 Algebraic Results

$$\text{if} \quad \frac{a}{b} = \frac{c}{d} \quad \text{then:}$$

$$ad = bc \qquad \text{diagonal product} \qquad (1.1)$$

$$\frac{a}{c} = \frac{b}{d} \qquad \text{diagonal exchange} \qquad (1.2)$$

$$\frac{b}{a} = \frac{d}{c} \qquad \text{inverse} \qquad (1.3)$$

$$\frac{a+c}{b} = \frac{c+d}{d} \qquad \text{addend} \qquad (1.4)$$

Factors of $x^n - a^n$

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1}) \qquad (1.5)$$

Sum and difference of two cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \qquad (1.6)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \qquad (1.7)$$

$$a^2 - b^2 = (a+b)(a-b) \qquad (1.8)$$

Expansions of various squares and cubes

$$(a+b+c+\cdots+n)^2 = a^2 + b^2 + c^2 + \cdots + n^2 + 2\sum ab$$

$$\text{(where } \sum ab \text{ represents all possible pairs of } a, b, c, \ldots, n) \qquad (1.9)$$

$$(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd \qquad (1.10)$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \qquad (1.11)$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \qquad (1.12)$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (1.13)$$

2 Absolute value

Definition 2.1 The definition of the absolute value of a is

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases} \qquad (2.1)$$

Some properties

$$\text{if } |x| = a \quad \text{then} \quad x = \pm a \tag{2.2}$$
$$\text{if } |x| < a \quad \text{then} \quad -a < x < a \tag{2.3}$$
$$\text{if } |x| > a \quad \text{then} \quad x < -a \text{ or } x > a \tag{2.4}$$

3 Inequalities

If $a > b$, $c > d$, then:

$$a \pm c > b \pm c \tag{3.1}$$
$$ac > bc \qquad (c > 0) \tag{3.2}$$
$$ac < bc \qquad (c < 0) \tag{3.3}$$
$$ac > bd \qquad (a, b, c, d \text{ all } > 0) \tag{3.4}$$
$$a^2 > b^2 \qquad (a, b \text{ both } > 0) \tag{3.5}$$
$$\frac{1}{a} < \frac{1}{b} \qquad (a, b \text{ both } > 0) \tag{3.6}$$

4 Trigonometry

"All Stations To Central":

Reciprocal ratios

$$\csc\theta = \frac{1}{\sin\theta} \tag{4.1}$$
$$\sec\theta = \frac{1}{\cos\theta} \tag{4.2}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \tag{4.3}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{4.4}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{4.5}$$
$$\tan^2\theta + 1 = \sec^2\theta \tag{4.6}$$
$$1 + \cot^2\theta = \csc^2\theta \tag{4.7}$$

Inverse formulas

$$\sin^{-1} x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} \qquad x \neq 1 \tag{4.8}$$
$$\cos^{-1} x = \tan^{-1}\frac{\sqrt{1-x^2}}{x} \qquad x \neq 0 \tag{4.9}$$

Co-ratios—(complementary angles)

$$\sin(90^\circ - \theta) = \cos\theta \qquad \cos(90^\circ - \theta) = \sin\theta \tag{4.10}$$
$$\tan(90^\circ - \theta) = \cot\theta \qquad \cot(90^\circ - \theta) = \tan\theta \tag{4.11}$$
$$\sec(90^\circ - \theta) = \csc\theta \qquad \csc(90^\circ - \theta) = \sec\theta \tag{4.12}$$

$$\sin(180^\circ - \theta) = \sin\theta \qquad \sin(180^\circ + \theta) = -\sin\theta \tag{4.13}$$
$$\cos(180^\circ - \theta) = -\cos\theta \qquad \cos(180^\circ + \theta) = -\cos\theta \tag{4.14}$$
$$\tan(180^\circ - \theta) = -\tan\theta \qquad \tan(180^\circ + \theta) = \tan\theta \tag{4.15}$$

$$\sin(360^\circ - \theta) = \sin -\theta = -\sin\theta \qquad \text{(odd)} \tag{4.16}$$
$$\cos(360^\circ - \theta) = \cos -\theta = \cos\theta \qquad \text{(even)} \tag{4.17}$$
$$\tan(360^\circ - \theta) = \tan -\theta = -\tan\theta \qquad \text{(odd)} \tag{4.18}$$

Exact values

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} \tag{4.19}$$
$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2} \tag{4.20}$$
$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2} \tag{4.21}$$
$$\tan 45^\circ = \cot 45^\circ = 1 \tag{4.22}$$
$$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}} \tag{4.23}$$
$$\tan 60^\circ = \cot 30^\circ = \sqrt{3} \tag{4.24}$$
$$\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{4} \tag{4.25}$$
$$\sin 75^\circ = \cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} \tag{4.26}$$
$$\tan 15^\circ = \cot 75^\circ = 2-\sqrt{3} \tag{4.27}$$
$$\tan 75^\circ = \cot 15^\circ = 2+\sqrt{3} \tag{4.28}$$

Addition formulas

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi \tag{4.29}$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi \tag{4.30}$$
$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi} \tag{4.31}$$
$$\cot(\theta \pm \phi) = \frac{\cot\theta\cot\phi \mp 1}{\cot\theta \pm \cot\phi} \sin 2\theta \qquad = 2\sin\theta\cos\theta \tag{4.32}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{4.33}$$
$$= 1 - 2\sin^2\theta \tag{4.34}$$
$$= 2\cos^2\theta - 1 \tag{4.35}$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{4.36}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{4.37}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \tag{4.38}$$

Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{4.39}$$

Area of a triangle

$$\text{Area of any triangle } = \frac{ab\sin C}{2} \tag{4.40}$$

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{(side formula)} \tag{4.41}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \text{(angle formula)} \tag{4.42}$$

"Little t" formula

$$t = \tan\frac{x}{2} \tag{4.43}$$

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} \tag{4.44}$$

$$= \frac{2t}{1-t^2} \tag{4.45}$$

$$\sin x = \frac{2t}{1+t^2} \tag{4.46}$$

$$\cos x = \frac{1-t^2}{1+t^2} \tag{4.47}$$

$$\text{if } t = \tan\frac{x}{2} \text{ then } dx = \frac{2\,dt}{1+t^2} \tag{4.48}$$

Period and amplitude

$$y = a\sin(\omega t + \phi) \qquad \text{amplitude} = a,\ \text{period} = \frac{2\pi}{\omega} \tag{4.49}$$

$$y = a\cos(\omega t + \phi) \qquad \text{amplitude} = a,\ \text{period} = \frac{2\pi}{\omega} \tag{4.50}$$

$$y = a\tan(\omega t + \phi) \qquad \text{amplitude} = \infty,\ \text{period} = \frac{\pi}{\omega} \tag{4.51}$$

Special limits

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \tag{4.52}$$

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1 \tag{4.53}$$

The 3θ results

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{4.54}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \tag{4.55}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \tag{4.56}$$

4.1 Sum and product formulas

$$\sin(\theta+\phi) + \sin(\theta-\phi) = 2\sin\theta\cos\phi \tag{4.57}$$

$$\sin(\theta+\phi) - \sin(\theta-\phi) = 2\sin\phi\cos\theta \tag{4.58}$$

$$\cos(\theta+\phi) + \cos(\theta-\phi) = 2\cos\theta\cos\phi \tag{4.59}$$

$$\cos(\theta+\phi) - \cos(\theta-\phi) = -2\sin\theta\sin\phi \tag{4.60}$$

$$\text{if} \begin{cases} \angle_1 = \theta + \phi \\ \angle_2 = \theta - \phi \end{cases} \quad \text{then} \quad \begin{array}{c} \theta = \frac{1}{2}(\angle_1 + \angle_2) \\ \phi = \frac{1}{2}(\angle_1 - \angle_2) \end{array} \quad \text{use } \theta > \phi \tag{4.61}$$

Using these formulas:

$$\sin a = \frac{e^{ia} - e^{-ia}}{2i} \tag{4.62}$$

$$\cos a = \frac{e^{ia} + e^{-ia}}{2} \tag{4.63}$$

it is not too hard to derive the sum and product formulas, e.g.,

$$\begin{aligned} \cos a \cos b &= \tfrac{1}{4}\left(e^{ia} + e^{-ia}\right)\left(e^{ib} + e^{-ib}\right) \\ &= \tfrac{1}{4}\left(e^{i(a+b)} + e^{i(a-b)} + e^{-i(a-b)} + e^{-i(a+b)}\right) \\ &= \tfrac{1}{4}\left(e^{i(a+b)} + e^{-i(a+b)}\right) + \tfrac{1}{4}\left(e^{i(a-b)} + e^{-i(a-b)}\right) \\ &= \tfrac{1}{2}\cos(a+b) + \tfrac{1}{2}\cos(a-b). \end{aligned} \tag{4.64}$$

4.2 Sum of two waves

$$a\cos\omega t + b\sin\omega t = \sqrt{a^2 + b^2}\,\cos\left(\omega t - \tan^{-1}\frac{b}{a}\right) \tag{4.65}$$

Here we use the fact that

$$a + jb = \sqrt{a^2 + b^2}\, e^{j\tan^{-1}\frac{b}{a}} \tag{4.66}$$

and

$$a - jb = \overline{a + jb} = \sqrt{a^2 + b^2}\, e^{-j\tan^{-1}\frac{b}{a}} \tag{4.67}$$

$$\tag{4.68}$$

$$a\cos\omega t + b\sin\omega t = \frac{a}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) + \frac{b}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \tag{4.69}$$

$$= \frac{a}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) - \frac{jb}{2}\left(e^{j\omega t} - e^{-j\omega t}\right) \tag{4.70}$$

$$= \frac{1}{2}\left(e^{j\omega t}(a - jb) + e^{-j\omega t}(a + jb)\right) \quad \text{(from (4.66) and (4.67))} \tag{4.71}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}\left(e^{j\omega t}e^{-j\tan^{-1}\frac{b}{a}} + e^{-j\omega t}e^{j\tan^{-1}\frac{b}{a}}\right) \tag{4.72}$$

$$= \frac{\sqrt{a^2 + b^2}}{2}\left(e^{j(\omega t - \tan^{-1}\frac{b}{a})} + e^{-j(\omega t - \tan^{-1}\frac{b}{a})}\right) \tag{4.73}$$

$$= \sqrt{a^2 + b^2}\cos\left(\omega t - \tan^{-1}\frac{b}{a}\right) \tag{4.74}$$

Period of sum of two sine waves: If $\cos\omega_1 t + \cos\omega_2 t$ is periodic with period T, then $\exists m, n \in \mathbb{Z} : \omega_1 T = 2\pi m, \omega_2 T = 2\pi n$, i.e., $\dfrac{\omega_1}{\omega_2} = \dfrac{m}{n}$ is rational.

Also, to find T:
$\omega_2 = \dfrac{n}{m}\omega_1, \quad T = \dfrac{2\pi}{\omega_1}m = \dfrac{2\pi}{\omega_2}n.$

The same is true of $\cos(\omega_1 t + \theta), \cos(\omega_2 t + \phi), \quad \theta, \phi \in \mathbb{R}$, i.e., phase doesn't affect the value of T here.

5 Coordinate geometry: straight lines

Equations of straight lines The equation of a line $\parallel$ y axis through (a, b) is

$$y = b \tag{5.1}$$

The equation of a line $\parallel$ x axis through (a, b) is

$$x = a \tag{5.2}$$

The equation of a line with slope m and y intercept b is

$$y = mx + b \tag{5.3}$$

The equation of a line with slope m through (x_1, y_1) is

$$y - y_1 = m(x - x_1) \tag{5.4}$$

The equation of a line through (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{5.5}$$

The equation with x intercept a and y intercept b is

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{5.6}$$

This "general form" of equation for a line has slope $= \frac{-A}{B}$, x intercept $= \frac{-C}{A}$, y intercept $= \frac{-C}{B}$:

$$Ax + By + C = 0 \tag{5.7}$$

Distance and point formulas

The distance between (x_1, x_2) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{5.8}$$

The distance of (x_1, y_1) from $Ax + By + C = 0$ is

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right| \tag{5.9}$$

The coordinates of the point which divides the join of $Q = (x_1, y_1)$ and $R = (x_2, y_2)$ in the ratio $m_i : m_2$ is

$$P = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right) \tag{5.10}$$

From Q to P to R. If P is outside of the line QR, then QP, RP are measured in opposite senses, and $\frac{m_1}{m_2}$ is negative.

The coordinates of the midpoint of the join of (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \tag{5.11}$$

The angle α between two lines with slopes m_1, m_2 is given by

$$\tan\alpha = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right| \qquad \text{for } \alpha \text{ acute} \tag{5.12}$$

$$\tan\alpha = -\left| \frac{m_2 - m_1}{1 + m_2 m_1} \right| \qquad \text{for } \alpha \text{ obtuse} \tag{5.13}$$

6 Logarithms and Indexes

Basic index laws

$$a^m \times a^n = a^{m+n} \tag{6.1}$$

$$a^m \div a^n = a^{m-n} \tag{6.2}$$

$$(a^m)^n = a^{mn} \tag{6.3}$$

$$a^0 = 1 \tag{6.4}$$

$$a^{-n} = \frac{1}{a^n} \tag{6.5}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \qquad (\text{if } a^m > 0) \tag{6.6}$$

$$(ab)^n = a^n b^n \tag{6.7}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \tag{6.8}$$

Definition 6.1 (Definition of a logarithm) if $\log_a x = y$ then $x = a^y$. The number a is called the *base* of the logarithm.

Change of base of logarithms

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{6.9}$$

Basic laws of logarithms

$$\log_a mn = \log_a m + \log_a n \tag{6.10}$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n \tag{6.11}$$

$$\log_a m^x = x \log_a m \tag{6.12}$$

$$a^x = e^{x \ln a} \tag{6.13}$$

$$x^e = e^{e \log x} \tag{6.14}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \forall x. \tag{6.15}$$

7 Series and Sequences

Arithmetic progression (AP): is of the form:

$$a, a+d, a+2d, \ldots, u_n$$

$$u_n = a + (a-1)d \tag{7.1}$$

Tests for arithmetic progression

$$u_2 - u_1 = u_3 - u_2 = \cdots = u_n - u_{n-1}$$

$$\text{if } a, b, c \text{ are sequential terms in AP, then } b = \frac{a+c}{2}$$

Sum of an AP

$$S_n = \sum_{k=1}^{n} \big(a + (k-1)d\big) \tag{7.2}$$

$$u_n = S_n - S_{n-1} \tag{7.3}$$

$$S_n = \frac{n(a+u_n)}{2} \qquad \text{if given last term } u_n \tag{7.4}$$

$$S_n = \frac{n\big(2a+(n-1)d\big)}{2} \qquad \text{if last term not given} \tag{7.5}$$

Geometric progression (GP): is of the form:

$$a, ar, ar^2, \ldots, ar^{n-1}$$

$$u_n = ar^{n-1} \tag{7.6}$$

Test for GP

$$\frac{u_2}{u_1} = \frac{u_3}{u_2} = \cdots = \frac{u_n}{u_{n-1}}$$

$$u_2^2 = u_1 u_3, \ldots, u_{n-1}^2 = u_{n-2} u_n$$

Sum of GP

$$S_n = a \sum_{k=1}^{n} r^{k-1}$$

$$S_n = \frac{a(r^n - 1)}{r-1} \qquad \text{use if } |r| > 1 \tag{7.7}$$

$$S_n = \frac{a(1 - r^n)}{1-r} \qquad \text{use if } |r| < 1 \tag{7.8}$$

Sum to infinity

$$S_\infty = \frac{a}{1-r} \qquad \text{where } |r| < 1. \tag{7.9}$$

$$\begin{aligned}\sum_{k=1}^{n} r^k &= r\sum_{k=1}^{n} r^{n-1} \\ &= \frac{r(r^n-1)}{r-1} \\ &= \frac{r^{n+1}-r}{r-1}\end{aligned} \tag{7.10}$$

Some other useful sums

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7.11}$$

$$\sum_{i=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \tag{7.12}$$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \tag{7.13}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{7.14}$$

$$a^x = e^{x\ln a} = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \dots \tag{7.15}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \qquad -1 < x \le 1. \tag{7.16}$$

Series relevant to $\mathcal{Z}$ transforms

$$\sum_{n=0}^{M} z^n = 1 + z + z^2 + z^3 + \cdots + z^{M-1} + z^M \tag{7.17}$$

$$z \sum_{n=0}^{M} z^n = z + z^2 + z^3 + \cdots + z^{M-1} + z^{M+1} \qquad \text{multiplying b.s. by } z \tag{7.18}$$

$$(1 - z) \sum_{n=0}^{M} z^n = 1 - z^{M+1} \qquad \text{subtracting (7.17) − (7.18)} \tag{7.19}$$

$$\therefore \sum_{n=0}^{M} z^n = \frac{1 - z^{M+1}}{1 - z} \tag{7.20}$$

Now

$$\sum_{n=0}^{\infty} z^n = \lim_{M\to\infty} \sum_{n=0}^{M} z^n \tag{7.21}$$

$$= \lim_{M\to\infty} \frac{1 - z^{M+1}}{1 - z} \tag{7.22}$$

$$= \frac{1}{1 - z} \quad \forall z : |z| < 1 \tag{7.23}$$

This is from Thomas & Finney, pages 610–611.

8 Finance

Simple interest

$$I = Ptr \tag{8.1}$$

where P = principal, t = time, r = rate (i.e., 6% $\longrightarrow \frac{6}{100}$)

Compound interest

$$A = P(1 + r)^n \tag{8.2}$$

where P = principal, n = number of time periods, r = rate per time period, A = amount you get.

Reducible interest: This uses $S_n = \frac{a(r^n-1)}{r-1}$ where a is M, $r = 1 + \frac{R}{100}$.

$$\frac{M\left((1+\frac{R}{100})^n - 1\right)}{\frac{R}{100}} = P\left(1 + \frac{R}{100}\right)^n \tag{8.3}$$

$$\text{i.e., } M = \frac{P(1+\frac{R}{100})^n \cdot \frac{R}{100}}{(1+\frac{R}{100})^n - 1} \tag{8.4}$$

where M = equal installment paid per unit time
R = % rate of interest per unit time (Note: same unit of time as for M)
n = number of installments.

Superannuation

$$S_n = \frac{A\Big(1 + \frac{R}{100}\left((1+\frac{R}{100})^n - 1\right)\Big)}{\frac{R}{100}} \tag{8.5}$$

where A is an equal installment per unit time. n, R are as above.

9 Quadratic equations

The solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{9.1}$$

10 Differentiation

From first principles:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{10.1}$$

$$\text{or } f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{10.2}$$

$$\frac{d}{dx} x^n = nx^{n-1} \tag{10.3}$$

$$\frac{d}{dx} c = 0 \tag{10.4}$$

$$\frac{d}{dx} c\, f(x) = c\, f'(x) \tag{10.5}$$

$$\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x) \tag{10.6}$$

Product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{10.7}$$

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx} \tag{10.8}$$

Quotient rule

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \tag{10.9}$$

Chain rule (Function of a function rule, composite function rule, substitution rule)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{10.10}$$

10.1 Derivatives of trig functions

$$\frac{d}{dx}\sin f(x) = f'(x)\cos f(x) \tag{10.11}$$

$$\frac{d}{dx}\cos f(x) = -f'(x)\sin f(x) \tag{10.12}$$

$$\frac{d}{dx}\tan f(x) = f'(x)\sec^2 f(x) \tag{10.13}$$

$$\frac{d}{dx}\sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1-f(x)^2}} \qquad -\frac{\pi}{2} \le \sin^{-1} f(x) \le \frac{\pi}{2} \tag{10.14}$$

$$\frac{d}{dx}\cos^{-1} f(x) = \frac{-f'(x)}{\sqrt{1-f(x)^2}} \qquad 0 \le \cos^{-1} f(x) \le \pi \tag{10.15}$$

$$\frac{d}{dx}\tan^{-1} f(x) = \frac{f'(x)}{1+f(x)^2} \qquad -\frac{\pi}{2} \le \tan^{-1} f(x) \le \frac{\pi}{2} \tag{10.16}$$

$$\frac{d}{dx}\sec x = \sec x \tan x \tag{10.17}$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \tag{10.18}$$

$$\frac{d}{dx}\cot x = -\csc^2 x \tag{10.19}$$

$$\frac{d}{dx}\sinh x = \cosh x \tag{10.20}$$

$$\frac{d}{dx}\cosh x = \sinh x \tag{10.21}$$

10.2 Derivatives of exponential and log functions

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)} \tag{10.22}$$

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)} \tag{10.23}$$

$$\frac{d}{dx}e^{x} = e^{x} \tag{10.24}$$

$$\frac{d}{dx}a^{x} = \ln a \cdot a^{x} \qquad \text{since } a^k = e^{k\ln a} \tag{10.25}$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln y\,\frac{dy}{dx} \tag{10.26}$$

$$\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx} \tag{10.27}$$

10.3 Stationary points

$f'(x) = 0$ at a stationary point of $f(x)$. (10.28)

For a stationary point $(w, f(w))$:

if gradient changes sign around $(w, f(w))$, local maximum or minimum (10.29)

if gradient does not change sign around $(w, f(w))$, point of inflection. (10.30)

if $f''(w) > 0$ then $(w, f(w))$ is a minimum point. (10.31)

if $f''(w) < 0$ then $(w, f(w))$ is a maximum point. (10.32)

if $f''(w) = 0$ then it could be a minimum, maximum or point of inflection. (10.33)

11 Integration

11.1 Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \tag{11.1}$$

11.2 Numerical integration

Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\big(y_0+y_n+2(y_1+y_2+y_3+\cdots+y_{n-1})\big) \tag{11.2}$$

Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+4y_{n-1}+y_n) \qquad n \text{ is even} \tag{11.3}$$

11.3 Indefinite integrals

$$\int K\,dx = Kx + c \tag{11.4}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \tag{11.5}$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \tag{11.6}$$

$$\int \sin ax\,dx = \frac{-1}{a}\cos ax + c \tag{11.7}$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + c \tag{11.8}$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c \tag{11.9}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c \tag{11.10}$$

$$\int \frac{dx}{x} = \ln x + c \tag{11.11}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c \tag{11.12}$$

$$\int \frac{dx}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c \tag{11.13}$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c \tag{11.14}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + c \tag{11.15}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c \tag{11.16}$$

$$= \cos^{-1}\frac{x}{a} + c_2, \qquad -a < x < a \tag{11.17}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \tag{11.18}$$

$$\int e^u\,du = e^u + c, \quad \text{i.e.,} \int e^{f(x)} f'(x)\,dx = e^{f(x)} + c \tag{11.19}$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c \tag{11.20}$$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\frac{x-a}{a+a} + c \tag{11.21}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{a+x}{a-x} + c \tag{11.22}$$

$$\int \ln x\,dx = x\ln x - x + c \tag{11.23}$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \log(x + \sqrt{a^2 + x^2}) + c \tag{11.24}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2}) + c \quad \forall x : |x| > a \tag{11.25}$$

$$\int \tan x\,dx = \log\sec x + c \tag{11.26}$$

$$\int \sec x\,dx = \ln(\sec x + \tan x) + c \tag{11.27}$$

$$= -\ln(\sec x - \tan x) + c \tag{11.28}$$

$$= \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c \tag{11.29}$$

$$\int \cot x\,dx = \ln\sin x + c \tag{11.30}$$

$$\int \csc^2(ax)\,dx = -\frac{1}{a}\cot(ax) + c \tag{11.31}$$

$$\int a^x\,dx = \int e^{x\ln a}\,dx = \frac{e^{x\ln a}}{\ln a} + c = \frac{a^x}{\ln a} + c_2 \tag{11.32}$$

$$(\text{let } y = a^x, \text{ then } \ln y = x\ln a, \text{ so } y = e^{x\ln a}). \tag{11.33}$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + c \tag{11.34}$$

$$\int \sinh ax \, dx = \frac{1}{a} \cosh ax + c \tag{11.35}$$

Trig substitutions

$$\begin{aligned} \text{for} \quad & a^2 + x^2 && \text{try} && x = a \tan\theta && (11.36) \\ & a^2 - x^2 && \text{try} && x = a \sin\theta && (11.37) \\ & x^2 + a^2 && \text{try} && x = a \sec\theta && (11.38) \end{aligned}$$

$$\text{if } t = \tan\frac{x}{2} \text{ then } dx = \frac{2\,dt}{1+t^2} \tag{11.39}$$

11.4 Definite integrals

$$\frac{d}{dx}\left\{\int_a^0 f(t)\,dt\right\} = f(x) \tag{11.40}$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \tag{11.41}$$

12 Areas and volumes of geometric shapes

Length of an arc with angle θ and radius r is $\ell = r\theta$.

Area of a sector angle θ and radius r is $A = \frac{1}{2}r^2\theta$

Area of a segment of a circle with angle θ and radius r is $A = \frac{1}{2}r^2(\theta - \sin\theta)$

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2}r^2 \sin\theta \\ \therefore \text{area of segment} &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta \end{aligned}$$

Area of a triangle with height h and base b is $A = \frac{1}{2}hb$

Area of parallelogram with height h and base b is $A = bh$

Area of a trapezium with parallel sides of length a, b and height h is $A = \frac{1}{2}h(a+b)$

Area of a rhombus with diagonals of length a, b is $A = \frac{1}{2}ab$

Area of a circle is $A = \pi r^2$, circumference is $C = 2\pi r$

Area of an ellipse with semi-major axis length a, semi-minor axis length b is $A = \pi ab$. (Note that a, b become the radius as the ellipse approaches the shape of a circle.)

12.1 Volumes

In the following, h is the height of the shape, r is a radius.

Right rectangular prism or oblique rectangular prism with base lengths a and b: $V = abh$

Cylinder with circular base, both upright and oblique, $V = \pi r^2 h$

Pyramids with rectangular bases of side length a, b, both right pyramids and oblique pyramids: $V = \frac{1}{3}abh$

Cones: $V = \frac{1}{3}\pi r^2 h$

The curved surface area of an upright cone with length from apex to edge of the base s (s is not the height, but the length of the side of the cone): $S = \pi rs$.

Volume of a sphere $V = \frac{4}{3}\pi r^3$

Surface area of a sphere is $S = 4\pi r^2$

13 Simple harmonic motion

Definition 13.1

$$\ddot{x} = -n^2 x \qquad n \text{ is any constant} \tag{13.1}$$

Other properties

$$v^2 = n^2(a^2 - x^2) \qquad a = \text{amplitude} \tag{13.2}$$

$$x = a\cos nt \tag{13.3}$$

$$T = \frac{2\pi}{n} \qquad T \text{ is the period} \tag{13.4}$$

$$v_{\text{max}} = |na| \tag{13.5}$$

$$\ddot{x}_{\text{max}} = -n^2 a \tag{13.6}$$

14 Newton's method

If x_1 is a good approximation to $f(x) = 0$ then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{14.1}$$

is a better approximation.

15 Binomial

The coefficients in the expansion of $(a+b)^n$ where $n = 0, 1, 2, 3, \dots$ are given by Pascal's triangle.

15.1 Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} T_{r+1} \tag{15.1}$$

$$= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\, a^{n-r} b^r \tag{15.2}$$

$$T_{r+1} = {}^nC_r a^{n-r} b^r \tag{15.3}$$

$$\text{where } {}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{15.4}$$

Pascal's triangle relationship

$${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_{r-n} \tag{15.5}$$

Sum of coefficients

$$\sum_{r=0}^{n} {}^nC_r = 2^n \tag{15.6}$$

Symmetrical relationship

$${}^nC_r = {}^nC_{n-r} \tag{15.7}$$

16 Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{even: catenary} \tag{16.1}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{odd: } 1 \to 1\text{, onto.} \tag{16.2}$$

$$\frac{d}{dx} \cosh x = \sinh x \tag{16.3}$$

$$\frac{d}{dx} \sinh x = \cosh x \tag{16.4}$$

Identities

$$\cosh^2 x - \sinh^2 x = 1 \tag{16.5}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \tag{16.6}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{16.7}$$

17 Sets of numbers

subset	**notation**	**elements**
Complex numbers	$\mathbb{C}$	
Real numbers	$\mathbb{R}$	
Natural numbers (or whole numbers)	$\mathbb{N}$	$1, 2, 3, 4, 5, \ldots$
Integers	$\mathbb{Z}$	$\ldots, -2, -1, 0, 1, 2, 3, \ldots$
Rational numbers (or fractions)	$\mathbb{Q}$	$0, 1, 2, -1, \frac{1}{2}, \frac{3}{4}, \frac{5}{3}, -\frac{1}{2}, -\frac{3}{7}, \ldots$

Reference: K. G. Binmore, *Mathematical Analysis: a straightforward approach*, Cambridge, 1983.

Note that Gunter called the set of integers $\mathbb{J}$. He also referred to a set $\mathbb{R}*$, and called the set of rationals $\mathbb{Q}$ or $\mathbb{R}$. He included 0 in the set of Cardinals or natural numbers, $\mathbb{N}$.

18 Complex numbers

Definition of i

Definition 18.1

$$\sqrt{-1} = \pm i \tag{18.1}$$

Equality

Definition 18.2 If $a + ib = c + id$, then $a = c$ and $b = d$.

Mod-arg theorems

Theorem 18.1

$$|z_1 z_2| = |z_1| \cdot |z_2| \tag{18.2}$$

$$\arg(z_1 z_2) = \arg z1 + \arg z_2 \tag{18.3}$$

THEOREM 18.2

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{18.4}$$

$$\arg\frac{z_1}{z_2} = \arg z1 - \arg z_2 \tag{18.5}$$

Euler's formula

$$\cos\theta + i\sin\theta = \operatorname{cis}\theta = e^{i\theta} \tag{18.6}$$

Gunter called $\cos\theta + i\sin\theta$ $\quad$ $\operatorname{cis}\theta$.

de Moivre's theorem

THEOREM 18.3

$$\cos n\theta + i\sin n\theta = \operatorname{cis} n\theta = e^{in\theta} \tag{18.7}$$

Cube roots of unity: if $z^3 = 1$ then $z = 1$ or $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$. Each root is the square of the other.

Using de Moivre's theorem: Here we use

$$z^3 = 1 \implies |z^3| = 1 \implies |z|^3 = 1 \implies |z| = 1.$$

$$\begin{aligned}
\text{let } z &= r^3(\cos\theta + i\sin\theta)^3 && \text{Now } r = 1\text{, as above}\\
\text{further, } z &= \cos 3\theta + i\sin 3\theta && \text{(by de Moivre's Theorem)}\\
\therefore \cos 3\theta + i\sin 3\theta &= 1 + i0 &&\\
\therefore \cos 3\theta &= 1, \quad i\sin 3\theta = i0 &&\\
\therefore 3\theta &= 0, 2\pi, 4\pi, \ldots &&\\
\therefore \theta &= 0, \frac{2\pi}{3}, \frac{-2\pi}{3} && \text{since } \theta < |\pi|.\\
\text{Hence } z &= \operatorname{cis} 0, \operatorname{cis}\frac{2\pi}{3}, \operatorname{cis}\frac{-2\pi}{3} &&\\
\text{Now } \left(\operatorname{cis}\frac{2\pi}{3}\right)^2 &= \operatorname{cis}\frac{4\pi}{3} && \text{(de Moivre's Theorem)}\\
&= \operatorname{cis}\frac{-2\pi}{3} && = \text{other cube root of unity.}
\end{aligned}$$

$\therefore$ one complex root is the square of the other. $\square$

This same method is used to find the nth root of unity.

See section 18 on the preceding page for the meaning of $\operatorname{cis}\theta$.

Multiplying a complex number by i rotates the number by $\frac{\pi}{2}$ in a positive (anti-clockwise) sense:

$$z + \overline{z} = 2\Re(z) \tag{18.8}$$

$$z - \overline{z} = 2i\Im(z) \tag{18.9}$$

$$|z| = \sqrt{z \cdot \overline{z}} \tag{18.10}$$

$$\arg z^{-1} = \arg \overline{z} \tag{18.11}$$

$$\overline{i\omega} = -i\overline{\omega} \tag{18.12}$$

19 Polynomials

Definition 19.1 *Monic* means the leading coefficient $a_n = 1$.

Theorem 19.1 *Every polynomial over $\mathcal{F}$ can be factorised into monic factors of degree ≥ 1 and a constant*

Remainder theorem:

Theorem 19.2 *$A(x)$ is a polynomial over $\mathcal{F}$; $A(x) = A(a)$ when $A(x)$ is divided by $(x - a)$.*

The factor theorem

Theorem 19.3 *$(x - a)$ is a factor of $A(x)$ iff $A(a) = 0$*

Theorem of rational roots of polynomials

Theorem 19.4 *If $\frac{r}{s}$ is a zero of $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ $(a_n \neq 0)$, r, s are relatively prime and $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then s/a_n and r/a_0, i.e., $(sx - r)$ is a factor of $P(x)$.*

Fundamental theorem of algebra

Theorem 19.5 *Every $P(x)$ with coefficients over $\mathbb{R}*$, $\mathbb{R}$ or $\mathbb{C}$ has a root $(P(\alpha) = 0)$ for some complex number α.*

See section 17 on page 24 for the particular meaning of "$\mathbb{R}*$, $\mathbb{R}$" here.

THEOREM 19.6 *A polynomial $P(x)$ of degree $n > 0$ has exactly n zeros in field $\mathbb{C}$ and hence exactly n linear factors over $\mathbb{C}$.*

THEOREM 19.7 *If $\alpha = a + ib$ is a root of $P(x)$ (whose coefficients are real), then its complex conjugate, $\overline{\alpha} = a - ib$ is also a root.*

THEOREM 19.8 *If $P(x)$ has degree n with real coefficients, if n is odd there is always at least one real root.*

19.1 Multiple roots & derived polynomials

THEOREM 19.9 *If the factor $(x - \alpha)$ occurs r times then we say it is an r-fold root, or α has a multiplicity of r.*

THEOREM 19.10 *If $P(x)$ has a root of multiplicity m then $P'(x)$ has a root of multiplicity $(m - 1)$.*

THEOREM 19.11 *If $P(x)$ over $\mathcal{F}$ is irreducible over $\mathcal{F}$ and $\deg P(x) > 1$, then there are no roots of $P(x)$ over $\mathcal{F}$.*

THEOREM 19.12 *2 different polynomials $A(x)$ and $B(x)$ over a non-finite field $\mathcal{F}$ cannot specify the same function in $\mathcal{F}$.*

THEOREM 19.13 *If two polynomials of the nth degree over a field $\mathcal{F}$ specify the same function for more than n elements of $\mathcal{F}$, then the two polynomials are equal.*

THEOREM 19.14 *If $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ over $\mathcal{F}$ where $n \neq 0$ and if $P(x)$ is completely reducible to n linear factors over $\mathcal{F}$ (i.e., if $P(x) = a_n(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $P(x)$), then*

$$\sum \alpha = \frac{-\alpha_{n-1}}{a_n} \tag{19.1}$$

$$\sum \alpha\beta = \frac{\alpha_{n-2}}{a_n} \tag{19.2}$$

$$\sum \alpha\beta\gamma = \frac{-\alpha_{n-3}}{a_n} \tag{19.3}$$

$$\vdots \tag{19.4}$$

$$\sum \ldots \alpha\beta\gamma \ldots \omega = (-1)^n \frac{\alpha_0}{a_n} \tag{19.5}$$

Also

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta \tag{19.6}$$

Definition of a rational function:

Definition 19.2 If $P(x)$, $Q(x)$ are 2 polynomials over $\mathcal{F}$, then $\frac{P(x)}{Q(x)}$ is a rational function over $\mathcal{F}$.

Definition of sum and product of rational functions:

Definition 19.3

$$\frac{A(x)}{B(x)} + \frac{C(x)}{D(x)} = \frac{A(x)D(x) + C(x)B(x)}{B(x)D(x)} \tag{19.7}$$

$$\frac{A(x)}{B(x)} \cdot \frac{C(x)}{D(x)} = \frac{A(x)C(x)}{B(x)D(x)} \tag{19.8}$$

19.2 Partial fractions

Theorem 19.15 *If F, P, Q are relatively prime polynomials over $\mathcal{F}$ and $\deg F < \deg PQ$ then we can find unique polynomials A, B such that*

$$\frac{F}{PQ} = \frac{A}{P} + \frac{B}{Q} \qquad \deg A < \deg P, \quad \deg B < \deg Q. \tag{19.9}$$